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1981 J. Phys. A: Math. Gen. 14 1629

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J. Phys. A: Math. Gen. **14** (1981) 1629–1640. Printed in Great Britain

Quaternion quantum mechanics as a true 3+1-dimensional theory of tachyons

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Received 8 March 1979, in final form 3 October 1980

Abstract. The new approach to quaternion quantum mechanics is given. It is shown that such a theory describes tachyons and that the quantum theory of tachyons should be a quaternionic one. This theory needs fundamental changes in basic physical assumptions and in a sense it is complementary to known physics. An important analogy between the basic notions of quaternionic and complex quantum mechanics emerges from the new scheme. The consequences for the theory of quantum and classical tachyons seem to be very important.

1. Introduction

Quaternion quantum mechanics (QM) was investigated by Finkelstein *et al* (1962, 1963) and Edmonts (1972, 1977). These authors usually began from the formal structure of QM based on the quaternion Hilbert space of states and from the idea of the system evolving in time. Our approach begins with the notion of the De Broglie wave. In this way we have discovered that the quaternion QM is, in fact, the theory of tachyons (Souček 1979).

There is a deep analogy between quaternion and complex QM based on the formal similarity between the operator $\mathrm{i}\partial_0$ of the infinitesimal time translation and the quaternion operator $\mathbf{i}\boldsymbol{\partial} = \mathrm{i}_1\partial_1 + \mathrm{i}_2\partial_2 + \mathrm{i}_3\partial_3$ (for the notation see § 2). The quaternion equation $\mathbf{i}\boldsymbol{\partial}\varphi = H(t)\psi$ is the analogue of the usual Schrödinger equation $\mathrm{i}\partial_0\varphi = H(\boldsymbol{x})\varphi$. This analogy is very broad—it covers notions, principles, equations and their solutions.

The interrelation between tachyons and quaternion QM seems to be fundamental. This theory of tachyons differs substantially from usual theories (Antippa and Everett 1973, Feinberg 1967, 1978, Schroer 1971). Besides the formal (aesthetic) reasons there are other reasons suggesting that our theory of tachyons should be a true one (problems of Cauchy data, of localisation etc). In the 1 + 1-dimensional space–time the problem is rather simple because of the possibility of making the (superluminal) transformation $x' = t$, $t' = x$ (Parker 1969, Recami and Mignani 1974, Vyšín 1977a, b). In the 3 + 1-dimensional case the problem is difficult because of different dimensionality of time and space and the use of quaternions is essential. Our theory brings rather *drastic consequences concerning the general concept of a tachyon* (see § 4).

In § 2 the quaternion Dirac equation is 'derived' and the time–space duality developed through subsequent sections is also suggested. These concern the non-relativistic tachyons (§ 3), classical tachyons (§ 4) and relativistic ones (§ 5). The

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quaternion analogue of electrodynamics is studied in § 6. It is suggested in the last section that there is a deep relation among complex numbers, quaternions, dimensions of time and space and topology of the inside and the outside of the light cone. All this is perfectly reflected in the structure of complex and quaternion QM. Perhaps the true name of this paper should be 'the trip into the world lying outside the light cone'.

2. The concept of quaternion quantum mechanics

The following notation for quaternions will be used

$$q = q^0 + \mathrm{i}_1 q^1 + \mathrm{i}_2 q^2 + \mathrm{i}_3 q^3 = q^0 + \mathbf{iq} \qquad q^+ = q^0 - \mathbf{iq}$$
$$|q|^2 = q^+ q = q^{02} + |\boldsymbol{q}|^2$$
$$\mathrm{i}_1^2 = -1 \qquad \mathrm{i}_1\mathrm{i}_2 = -\mathrm{i}_2\mathrm{i}_1 = \mathrm{i}_3, \ldots, q^0, \ldots, q^3 \in \mathbb{R}.$$

The goniometric form of a quaternion $q \in Q$ is

$$q = |q| \exp(\mathbf{i}\boldsymbol{n}\alpha) = |q|(\cos\alpha + \mathbf{i}\boldsymbol{n} \sin\alpha)$$
$$|\boldsymbol{n}| = 1 \qquad \alpha \in \mathbb{R}$$

where the imaginary quaternion unit $\mathbf{i}\boldsymbol{n}$ will be called the colour of the quaternion q. Our space–time notation will be

$$x = (x_0, \boldsymbol{x}) = (t, \boldsymbol{x}) \qquad \partial_0 = \partial/\partial x_0 \qquad \boldsymbol{\partial} = \partial/\partial \boldsymbol{x}$$

and we put $\hbar = c = 1$.

To our mind the previous investigations did not discover the deep meaning of the quaternion QM. The simplest way to see this is to consider the main problem of these investigations: which quaternion unit $\mathbf{i}\boldsymbol{n}$ should be put into the Schrödinger equation $\mathbf{i}\boldsymbol{n}\partial_0\psi = H\psi$? If we put $\mathbf{i}\boldsymbol{n}$ into it we lose the inner quaternion invariance, which means that all the i are equivalent. To take care of this we start from the very beginning of QM, i.e. from the De Broglie particle–wave dualism. The quaternion plane wave should be translation covariant. Thus the space–time translation should lead to the multiplication of it by an unit quaternion

$$\partial_\mu\psi = a_\mu\psi \qquad a_\mu = \mathbf{i}\boldsymbol{n}_\mu k_\mu \qquad |\boldsymbol{n}_\mu| = 1 \qquad \mu = 0, \ldots, 3. \tag{2.1}$$

It follows from one of these equations (μ fixed) that $\psi = \exp(\mathbf{i}\boldsymbol{n}_\mu k_\mu x_\mu)\psi'$ with ψ' independent of x_μ and thus all equations (2.1) together imply that the quaternions a_μ, $\mu = 0, \ldots, 3$ commute. Changing signs of the k_μ, if necessary, we have $\boldsymbol{n}_\mu = \boldsymbol{n}$, $\mu = 0, \ldots, 3$ and thus

$$\psi = \exp[\mathbf{i}\boldsymbol{n}(k_0 x_0 - \boldsymbol{kx})]. \tag{2.2}$$

We interpret the wave (2.2) as the wave describing a particle having 4-momentum k_μ and colour $\mathbf{i}\boldsymbol{n}$.

The natural candidate for the Schrödinger equation, $\mathbf{i}\boldsymbol{n}\partial_0\psi = -(1/2m)\boldsymbol{\partial}^2\psi$, cannot be accepted, because it specifies the colour $\mathbf{i}\boldsymbol{n}$ of its solutions in advance! To find the correct form of the equation we shall leave the problem for a moment and we shall look for the Dirac equation. Instead of four by four γ matrices, we need two by two quaternion matrices (because there are already three anticommuting quantities in Q).

The quaternion Dirac equation is

$$\mathbf{i}\boldsymbol{\partial}\psi = (P\partial_0 + \sigma_3 m)\psi \qquad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in Q^2 \qquad P = -\mathrm{i}\sigma_2. \tag{2.3}$$

The corresponding Klein–Gordon equation $(\partial_0^2 - \boldsymbol{\partial}^2 - m^2)\psi = 0$ is the equation for tachyons, and equation (2.3) *cannot* be converted into the bradyonic Dirac equation because there is no appropriate imaginary unit in Q (the units i_1, i_2, i_3 have already been used).

The equation (2.3) has already some 'Hamiltonian' form $\mathbf{i}\boldsymbol{\partial}\psi = H\psi$. This somewhat bizarre interpretation will be confirmed as a true one in the following sections. The discovered analogy between $\mathrm{i}\partial_0\psi = H\psi$ in complex QM and $\mathbf{i}\boldsymbol{\partial}\psi = H\psi$ in quaternion QM can be generalised to the more complete analogy called the time–space duality (TSD). Its meaning is the exchange of the roles of time and space if we go from complex numbers to quaternions. Thus in the quaternion QM we are going to consider the space variable as an evolutionary one (the state of a system being given in the time variable) and to show, moreover, that such an interpretation is natural and necessary in the theory of tachyons.

Now we shall show that the quaternion Dirac equation (2.3) is in fact the usual Dirac equation written for tachyons. Let us start with one concrete representation of the Dirac equation

$$\mathrm{i}\partial_0\varphi = (\mathrm{i}\sigma_2 \otimes \boldsymbol{\sigma}\boldsymbol{\partial} + \sigma_1 \otimes \mathbb{1} m)\varphi = H_{\mathrm{B}}\varphi \tag{2.4}$$

where

$$\sigma_2 \otimes \boldsymbol{\sigma} = \begin{pmatrix} 0 & -\mathrm{i}\boldsymbol{\sigma}\\ \mathrm{i}\boldsymbol{\sigma} & 0\end{pmatrix} \qquad \varphi \in C^4 \qquad m = \mathrm{i}\mu \qquad \mu > 0.$$

The Hamiltonian H_{B} *is not the Hermitian operator* and thus the usual interpretation of this equation fails. The only way to save the Hermitian properties of the Hamiltonian is to make the evolution operator from the operator $\mathrm{i}\sigma_3 \otimes \boldsymbol{\sigma}\boldsymbol{\partial}$. Multiplying (2.4) by $\sigma_3 \otimes \mathbb{1}$ we obtain

$$\mathbb{1} \otimes (-\mathrm{i}\boldsymbol{\sigma})\boldsymbol{\partial}\varphi = [P \otimes \mathbb{1}\partial^0 + \sigma_3 \otimes \mathbb{1}\mu]\varphi = H_{\mathrm{T}}\varphi \tag{2.5}$$

where H_{T} is found to be Hermitian. This equation can be identified with equation (2.3) if we introduce the quaternion notation. We can write

$$\begin{vmatrix}\varphi_1\\ \varphi_2\end{vmatrix} = (\mathrm{Re}\,\varphi_1 + \mathrm{i}\sigma_3\,\mathrm{Im}\,\varphi_1 - \mathrm{i}\sigma_2\,\mathrm{Re}\,\varphi_2 + \mathrm{i}\sigma_1\,\mathrm{Im}\,\varphi_2)\begin{vmatrix}1\\ 0\end{vmatrix}$$

with the analogical formula for the lower part of φ. Now all quantities in equation (2.5) are written by means of $\mathrm{i}\sigma_k$ and real numbers and it suffices to make the substitution

$$\begin{vmatrix}1\\ 0\end{vmatrix} \to 1 \qquad -\mathrm{i}\sigma_k \to \mathrm{i}_k \qquad \begin{vmatrix}\varphi_1\\ \varphi_2\end{vmatrix} \to \psi_1 \in Q \qquad \begin{vmatrix}\varphi_3\\ \varphi_4\end{vmatrix} \to \psi_2 \in Q.$$

We want to make a note on the meaning of the quaternion notation. The complex QM can be written in real numbers if we decompose each quantity into the real and the imaginary part. But then the basic invariance saying that vectors φ and $\varphi\,\mathrm{e}^{\mathrm{i}\alpha}$ denote the same state must be expressed as an additional explicit SO(2) invariance of this real QM. The same situation appears to be in quaternion QM: ψ and $\psi\,\mathrm{e}^{\mathbf{i}\boldsymbol{\alpha}}$ denote the same state. So the quaternion language is the way to express the natural invariance of the theory.

3. Non-relativistic quaternion QM

We saw in the second section that the quaternion QM can describe tachyons; we want to show now that this is, in fact, the only correct $3+1$-dimensional theory of tachyons. It was noticed (on the basis of the analogy argument) that the role of an evolution variable is played by the space one in the quaternion QM. To be more precise, the evolution variable cannot be chosen arbitrarily; the correct choice of it depends on the 'dynamics' of the considered system. To choose it correctly means to recognise well in which variable the initial data can be prescribed and in which variable we can localise the system.

It is well known (Feinberg 1978, Schroer 1971) that arbitrary initial values for tachyons cannot be prescribed on the hyperplane $\{t=0\}$ and that it is impossible to construct wave packets localised in space (see § 4). *Hence time cannot be the evolution variable for tachyons.* On the other hand it is possible and quite natural to prescribe arbitrary initial data on the time axis $\{\boldsymbol{x}=0\}$ (see this section) and to construct wave packets localised in time (see § 4). The reason for all this is the fact that the domain influenced by a tachyon lies outside the light cone, hence *the initial data on* $\{t=0\}$ *interfere with each other.* The best illustration is the case of the 'tachyon at rest' (the one with the velocity $v=\infty$). We have for it $E=0$, $p=m$.

The non-relativistic limit is now $v \gg 1$ and we obtain the dispersion relations

$$p=(m^2+E^2)^{1/2}\approx m(1+E^2/2m^2)=m+E^2/2m.$$

We saw that the quaternion Dirac equation can be obtained by the substitution $p\to \mathbf{i}\boldsymbol{\partial}$, $E\to P\partial_0$; hence the corresponding Schrödinger equation will become

$$\mathbf{i}\boldsymbol{\partial}\psi=-\frac{1}{2m}\partial_0^2\psi. \tag{3.1}$$

The plane-wave solutions have the form

$$\psi_{E,\boldsymbol{n}}=\exp[\mathbf{i}\boldsymbol{n}(Et-p_E\boldsymbol{nx})]A \qquad p_E=E^2/2m \qquad A\in Q. \tag{3.2}$$

Let us note that in the complex QM solutions of the Schrödinger equation have 'negative' time dependence $\varphi\sim\exp(-\mathrm{i}E_pt)$ (which is clearly related to the 1-dimensionality of time). It is difficult to find the space analogue of this property, but in quaternion QM we have

$$\psi\sim\exp(-\mathbf{i}\boldsymbol{n}p\boldsymbol{nx})$$

which can be interpreted as '*negative space*' *dependence* of the solution. The two cases $\exp(\pm\mathbf{i}\boldsymbol{n}p\boldsymbol{nx})$ have quite different character and cannot be transformed one to the other by the continuous change of the parameter $\boldsymbol{n}$ because ψ depends quadratically on $\boldsymbol{n}$. The fundamental importance of the relation $\boldsymbol{p}=p\boldsymbol{n}$ between the wavevector $\boldsymbol{p}$ of the solution and its colour should be noticed. The solution (3.2) can be interpreted as the '*positive 3-momentum' one* (the analogue of the positive energy solutions in complex QM). The other meaning is that the solution (3.2) is left-handed. (Let us note that the helicity has the invariant meaning only for massless particles; this could be explained by the fact that such particles can be considered as a limit case of tachyons.)

The set of all solutions (3.2) of equation (3.1) is parametrised by the energy E and the colour $\boldsymbol{n}$, hence the topological character of the set of parameters is $\mathbb{R}\times S_2/\{-1,1\}$ (here $S_2/\{-1,1\}$ is the two-dimensional projective space). It can be compared with the space $\mathbb{R}^3$ in the case of complex QM. The difference between these two sets of parameters is clearly related to the difference between the topology of the inside and the outside of the light cone. This fact is overlooked in all other theories of tachyons.

A general solution $\psi(\boldsymbol{x}, t)$ of equation (3.1) is a superposition of the solutions (3.2) with $A_{E,\boldsymbol{n}} = A_{-E,-\boldsymbol{n}}$. However, the functions $\psi_{E,\boldsymbol{n}}(0, t)$ are linearly dependent, hence we must prescribe something more on the time axis. To start with, let us decompose the function $\psi(\boldsymbol{x}, t)$

$$\psi(\boldsymbol{x}, t) = \int \mathrm{d}^2\boldsymbol{n}\, \tilde{\psi}(\boldsymbol{n}, \boldsymbol{nx}, t) \tag{3.3}$$

$$\tilde{\psi}(\boldsymbol{n}, x, t) = \int \mathrm{d}E \exp[\mathbf{i}\boldsymbol{n}(Et - p_E x)]A_{E,\boldsymbol{n}} \qquad x \in \mathbb{R}. \tag{3.4}$$

An arbitrary function $\tilde{\psi}(\boldsymbol{n}, 0, t)$ can be written uniquely in the form (3.4) with $x = 0$. Suitable initial data for equation (3.1) will be the system of functions

$$\{\tilde{\psi}(\boldsymbol{n}, 0, .)|\tilde{\psi}(-\boldsymbol{n}, 0, .) = \tilde{\psi}(\boldsymbol{n}, 0, .), \boldsymbol{n} \in S_2\}$$

defined on the time axis. The equation (3.1) for the function $\tilde{\psi}$ looks like

$$\mathbf{i}\boldsymbol{n}\,\partial_x\tilde{\psi}(\boldsymbol{n}, x, t) = -\frac{1}{2m}\,\partial_0^2\tilde{\psi}(\boldsymbol{n}, x, t) \qquad \boldsymbol{n} \in S_2. \tag{3.5}$$

The corresponding Green function formulae are

$$G(\boldsymbol{n}; x, t) = -\mathbf{i}\boldsymbol{n}\left(\frac{m}{2\pi\mathbf{i}\boldsymbol{n}x}\right)^{1/2} \exp(\mathbf{i}\boldsymbol{n}mt^2/2x)\theta(x)$$

$$\left(\mathbf{i}\boldsymbol{\partial} + \frac{1}{2m}\partial_0^2\right)G(\boldsymbol{n}; \boldsymbol{nx}, t) = \delta(t)\delta(\boldsymbol{nx})$$

$$\theta(\boldsymbol{nx})\tilde{\psi}(\boldsymbol{n}, \boldsymbol{nx}, t) = \mathbf{i}\boldsymbol{n}\int \mathrm{d}t'\, G(\boldsymbol{n}; \boldsymbol{nx}, t - t')\tilde{\psi}(\boldsymbol{n}, 0, t'). \tag{3.6}$$

The function $\tilde{\psi}(\boldsymbol{n}, \boldsymbol{nx}, t)$ can be interpreted as the '$\boldsymbol{n}$-colour' part of $\psi(\boldsymbol{x}, t)$ because it evolves in the $\boldsymbol{n}$ direction (i.e. depends only on $\boldsymbol{nx}$). Something like the space causality can be seen from relation (3.6): the $\boldsymbol{n}$-colour component $\tilde{\psi}(\boldsymbol{n}, \boldsymbol{nx}, t)$ propagates from the point $(0, t')$ only to the point $(\boldsymbol{x}, t)$ for which $\boldsymbol{nx} > 0$.

In the case of the Schrödinger equation with the potential

$$\mathbf{i}\boldsymbol{\partial}\psi = \left(-\frac{1}{2m}\partial_0^2 + V(t)\right)\psi \tag{3.7}$$

we shall suppose that the solution ψ can be written in the form (3.3). Then the equation (3.5) will take the form

$$\mathbf{i}\boldsymbol{n}\,\partial_x\tilde{\psi}(\boldsymbol{n}, x, t) = \left(-\frac{1}{2m}\partial_0^2 + V(t)\right)\tilde{\psi}(\boldsymbol{n}, x, t). \tag{3.8}$$

We see that the quaternion QM has the interesting $1+1$-dimensional character: the general solution is the sum of solutions $\tilde{\psi}(\boldsymbol{n}, x, t)$ of the one-dimensional equations (3.8). The more general potential $V(\boldsymbol{x}, t)$ would mix the $\boldsymbol{n}$-colour parts together.

The other thing which has to be changed in the quaternion QM is the concept of the conservation laws. It follows from $\partial_\mu j^\mu = 0$ in complex QM that

$$\partial_0 Q = \partial_0 \int \mathrm{d}^3x\, j^0 = -\int \mathrm{d}^3x\, \boldsymbol{\partial j} = -\oint \mathbf{ds}\, \boldsymbol{j} = 0.$$

The surface term equals zero, because the space-localised wave packets can be constructed. In quaternion QM only time-localised wave packets can be found (see § 4) and it forces us to define a 3-component quantity

$$Q_k(\boldsymbol{x}) = \int \mathrm{d}t\, j^k(\boldsymbol{x}, t) \qquad k = 1, 2, 3.$$

The corresponding law will be

$$\partial_k Q_k = \int \mathrm{d}t \partial_k j^k = -\int \mathrm{d}t \partial_0 j^0 = 0. \tag{3.9}$$

Now let us apply this general principle to the concept of the probability conservation. Using equation (3.7) and the conjugate one we obtain

$$j_k = \psi^+ \mathrm{i}_k \psi \qquad j_0 = \frac{1}{2m} \psi^+ \overleftrightarrow{\partial}_0 \psi$$
$$(a\overleftrightarrow{\partial} b = a\partial b - (\partial a) b) \tag{3.10}$$
$$\partial_0 j_0 + \partial_k j_k = 0 \qquad \partial_k Q_k = 0 \qquad Q_k = \int \mathrm{d}t\, j_k.$$

It is clear that the simple probability $\rho = \psi^+\psi$ is not conserved and moreover, the probability conservation law must be of the form (3.9). The current j_μ and the charges Q_k are purely imaginary quantities, so we put

$$j_\mu = \sum_{m=1}^{3} i_m j_{\mu m} \qquad Q_k = \sum_m i_m Q_{km} \qquad j_{\mu m}, Q_{km} \in \mathbb{R}.$$

The matrix $\Omega = (j_{km})_{k,m=1}^3$ then plays the role of the usual probability because of an apparent analogy between $i_m j_{km} = \psi^+ \mathrm{i}_k \psi$ and $\mathrm{i}\rho = \psi^* \mathrm{i} \psi$. Moreover, the matrix Ω is the *positive multiple* of an orthogonal matrix. In fact, for $\psi = |\psi| \exp(\boldsymbol{i n}\alpha)$ we have $\Omega = |\psi|^2 \exp(2\alpha N)$, where N is a matrix representing an infinitesimal rotation about the $\boldsymbol{n}$ axis in three-dimensional space.

Similarly for the quaternion Dirac equation (2.3) we obtain $j_k = \psi^+ \mathrm{i}_k \psi$, $j_0 = -\psi^+ P \psi$.

4. Classical tachyons

Let us consider a superposition of the solutions (3.2) of the equation (3.1) with parameters $(E, \boldsymbol{n})$ lying in intervals $|E - E_0| \leq \Delta E$, $|\boldsymbol{n} - \boldsymbol{n}_0| \leq \varepsilon$. The integration with respect to E yields

$$\psi(\boldsymbol{x}, t) = \int_{|\boldsymbol{n}-\boldsymbol{n}_0| \leq \varepsilon} \mathrm{d}^2\boldsymbol{n} \exp[\boldsymbol{in}(E_0 t - p_0 \boldsymbol{nx})]\, 2\frac{\sin(\Delta E(t - v_0 \boldsymbol{nx}))}{t - v_0 \boldsymbol{nx}} \tag{4.1}$$

where $p_0 = E_0^2/2m$, $v_0 = \mathrm{d}p_E/\mathrm{d}E = E_0/m$. The last term approximates to $\delta(t - v_0 \boldsymbol{nx})$. The integration with respect to $\boldsymbol{n}$ cannot improve the space localisation of the wave packet because ψ depends on $\boldsymbol{x}$ only via the term $\boldsymbol{nx}$. An example is provided by the tachyon with infinite velocity, where we have ($E_0 = p_0 = v_0 = 0$)

$$\psi(\boldsymbol{x}, t) = 2(\sin \Delta Et)/t \approx \delta(t).$$

The wavefunction (4.1) differs substantially from zero only on the hyperplane $t = \boldsymbol{wx}$, $\boldsymbol{w} = v_0\boldsymbol{n}$.

The classical theory of tachyons should be the classical limit of the corresponding quantum theory. But the impossibility of localising a tachyon (in quantum theory) leads us inevitably to the fact that *the classical trajectory of a tachyon should be a hyperplane* in space–time lying outside the light cone. Thus $\boldsymbol{x}$ should be considered as an independent variable and t as a dependent one describing the particle position. The velocity could be then expressed as $\boldsymbol{w} = \mathrm{d}t/\mathrm{d}\boldsymbol{x}$. It is easy to show that exactly *the same Einstein relation* is valid for the addition of such velocities. There is a duality between a 'bradyonic' trajectory $\{\boldsymbol{x} = \boldsymbol{v}x_0\}$ and a tachyonic one $\{y_0 = \boldsymbol{wy}\}$. This is given by the relation $x_\mu y^\mu = 0$. Such trajectories are dual if $\boldsymbol{v} = \boldsymbol{w}$.

At each time the best possible 'localisation' of a tachyon is not a point, but a two-dimensional plane in space, which is moving with superluminal velocity through space. Hence *tachyons do not have the usual particle-like manifestation.* If we admit the natural assumption, that the energy momentum of a tachyon is distributed over the two-dimensional plane, it is clear that a *tachyon cannot be detected with the aid of a usual apparatus* localised in space. From this point of view it is clear why the searching for tachyons had to fail. Moreover, an interaction between a localised bradyon and a tachyon extended over a 2-plane seems to be impossible. On the other hand we suggest that *tachyons will play an important role in the structure of the physical vacuum.*

5. Quaternion Dirac equation

The wave solution of the equation (2.3) can be written in the form ($|\boldsymbol{n}| = 1$)

$$\psi = \exp[-m^{-1}(E\sigma_1 - p_E \mathbf{i}\boldsymbol{n}\sigma_3)(Et - p_E\boldsymbol{nx})]\begin{pmatrix}A_1\\A_2\end{pmatrix}$$

where $p_E = (m^2 + E^2)^{1/2}$. The matrix in the exponent of this formula can be diagonalised, and in such a way we obtain the following basis

$$|E, \boldsymbol{n}, \varepsilon\rangle = \exp[-\varepsilon \mathbf{i}\boldsymbol{n}(Et - p_E\boldsymbol{nx})]A_{E,\boldsymbol{n}}^{(\varepsilon)} \qquad \varepsilon = \pm 1$$
$$A_{E,\boldsymbol{n}}^{(+1)} = \begin{vmatrix}1\\ \mathbf{i}\boldsymbol{n}E/(p_E+m)\end{vmatrix} \qquad A_{E,\boldsymbol{n}}^{(-1)} = \begin{vmatrix}-\mathbf{i}\boldsymbol{n}E/(p_E+m)\\ 1\end{vmatrix}. \tag{5.1}$$

Let us consider first the solutions with $E = 0$, i.e. the time-independent (or infinite velocity) ones: $|0, \boldsymbol{n}, \pm 1\rangle$, $\boldsymbol{n} \in S_2$. We interpret them as solutions with positive and negative 3-momentum respectively, because $\mathbf{i}\boldsymbol{\partial}|0, \boldsymbol{n}, \pm 1\rangle = \pm m|0, \boldsymbol{n}, \pm 1\rangle$. The set of all such solutions is thus divided into two connected components which are characterised by the 'positivity' and 'negativity' of their space dependence.

The situation is the same as in complex QM where we have $\psi_1, \psi_2 \sim \exp(-\mathrm{i}mt)$, $\psi_3, \psi_4 \sim \exp(\mathrm{i}mt)$. The general solutions (5.1) are eigenvectors of the 3-momentum operator $\mathbf{i}\boldsymbol{\partial}$

$$\mathbf{i}\boldsymbol{\partial}|E, \boldsymbol{n}, \varepsilon\rangle = p|E, \boldsymbol{n}, \varepsilon\rangle \qquad p = \varepsilon p_E \qquad \varepsilon = \pm 1.$$

So we have found a 'momentum' gap in the spectrum (analogous to the usual energy gap): $p \leqslant -m$ or $p \geqslant m$. Taking into account the identity $|-E, -\boldsymbol{n}, \varepsilon\rangle = |E, \boldsymbol{n}, \varepsilon\rangle$ we see that the topology of the set of all solutions is $\{\pm 1\} \times ((\mathbb{R} \times S_2)/\{\pm 1\})$.

The general solution

$$\psi = \sum_{E,\boldsymbol{n},\varepsilon} |E, \boldsymbol{n}, \varepsilon\rangle a_{E\boldsymbol{n}\varepsilon}, \; a_{E\boldsymbol{n}\varepsilon} = a_{-E,-\boldsymbol{n},\varepsilon} \in Q$$

can be written in the form

$$\psi(\boldsymbol{x}, t) = \sum_{\boldsymbol{n}} \tilde{\psi}(\boldsymbol{n}, \boldsymbol{n}\boldsymbol{x}, t)$$

$$\tilde{\psi}(\boldsymbol{n}, x, t) = \sum_{E,\varepsilon} \exp[-\varepsilon \mathbf{i}\boldsymbol{n}(Et - p_E x)] A^{(\varepsilon)}_{E,\boldsymbol{n}} a_{E\boldsymbol{n}\varepsilon}.$$

Then the quaternion Dirac equation transforms into the set of 1 + 1-dimensional Dirac equations

$$(\mathbf{i}\boldsymbol{n}\,\partial_x - P\partial_0 - m\sigma_3)\tilde{\psi}(\boldsymbol{n}, x, t) = 0 \qquad \boldsymbol{n} \in S_2.$$

The formal covariance of the quaternion Dirac equation can be easily found. Let $x_\mu \to x'_\mu$ be the rotation by an angle φ about the axis $\boldsymbol{n}$; then we have

$$\beta^\mu \partial_\mu = \exp(\mathbf{i}\boldsymbol{n}\varphi/2)\beta^\mu \partial'_\mu \exp(-\mathbf{i}\boldsymbol{n}\varphi/2)$$

where $\beta^0 = P$, $\beta^k = -\mathrm{i}_k \mathbb{1}$, $k = 1, 2, 3$. The covariance of the equation needs

$$\psi' = \exp(-\mathbf{i}\boldsymbol{n}\varphi/2)\psi.$$

Similarly for a boost with the velocity $v = th\varphi$ in the direction $\boldsymbol{n}$ we obtain

$$\beta^\mu \partial_\mu = \exp(-P\mathbf{i}\boldsymbol{n}\varphi/2)\beta^\mu \partial'_\mu \exp(-P\mathbf{i}\boldsymbol{n}\varphi/2)$$

$$\psi' = \exp(-P\mathbf{i}\boldsymbol{n}\varphi/2)\psi.$$

The existence of the momentum gap in the spectrum of equation (2.3) suggests the definition of the 'tachyonic' vacuum as a state in which all negative 3-momentum states $|E, \boldsymbol{n}, -1\rangle$, $E \in \mathbb{R}$, $\boldsymbol{n} \in S_2$ are occupied (assuming the Pauli exclusion principle) and the interpretation of a hole as an antiparticle. Contrary to other theories of tachyons, the vacuum in our theory is Lorentz invariant and stable with respect to the creation of pairs.

6. Quaternion electrodynamics

The Lagrangian

$$\mathscr{L}_0 = \tfrac{1}{2}\psi^+ \beta^\mu \overleftrightarrow{\partial}_\mu \psi + \psi^+ \sigma_3 m \psi \qquad \beta^0 = P \qquad \beta^k = -\mathrm{i}_k$$

(where we denoted $a\overleftrightarrow{\partial}b = a\partial b - (\partial a)b$) corresponding to equation (2.3) is real ($\beta^{\mu +} = -\beta^\mu$) and invariant with respect to the global phase transformations

$$\psi \to \psi a \qquad \psi^+ \to a^+ \psi^+ \qquad a = \mathrm{e}^{\mathrm{i}\alpha}.$$

To have a theory invariant with respect to local gauge transformations (where $a = a(x)$)

we must introduce a new imaginary field $A_\mu = \mathbf{i}\boldsymbol{A}_\mu$ with the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\psi^+\beta^\mu\psi\overleftarrow{D}_\mu - D^+_\mu\psi^+\beta^\mu\psi) + \psi^+\sigma_3 m\psi + \tfrac{1}{4}F^2_{\mu\nu}$$

$$D_\mu = \partial_\mu - gA_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$$

and transformation properties

$$A_\mu \to A'_\mu = a^+(x)A_\mu a(x) + \frac{1}{g}a^+(x)\partial_\mu a(x).$$

The equations of motion have, in our formalism, the form

$$\beta^\mu\psi(\overleftarrow{\partial}_\mu - gA_\mu) + m\sigma_3\psi = 0$$

$$\partial_\nu F^{\mu\nu} + g[A_\nu, F^{\mu\nu}] = gj^\mu \qquad j^\mu = \psi^+\beta^\mu\psi.$$

The conserved current is

$$J^\mu = j^\mu + j^\mu_{YM}, \qquad j^\mu_{YM} = -[A_\nu, F^{\mu\nu}]. \tag{6.1}$$

Thus the field A_μ is charged. The current $J^\mu(\mathbf{i}\boldsymbol{\alpha})$ generated by the infinitesimal transformation $a = 1 - \mathbf{i}\boldsymbol{\alpha}$ is related to J^μ by

$$J^\mu = \sum_{k=1}^{3} \mathrm{i}_k J^\mu(\mathrm{i}_k).$$

The potential $A_\mu = \mathbf{i}\boldsymbol{A}_\mu$ is exactly the well known SU(2) gauge field expressed in quaternion formalism, but the difference lies in its coupling to ψ. In our theory *the Yang–Mills field interacts only with one quaternion Dirac field*, while in the usual theory it interacts with an iso-doulet of Dirac fields. In fact, our potential A_μ acts on the same indices as the γ matrices do. The analogy with the usual electrodynamics is complete: $\mathrm{i}A_\mu \in C \to \mathbf{i}\boldsymbol{A}_\mu \in Q$ and $(\psi \to \psi\, \mathrm{e}^{\mathrm{i}\alpha}) \to (\psi \to \psi\, \mathrm{e}^{\mathbf{i}\boldsymbol{\alpha}})$; both are the natural phase invariances of the respective theories.

Let us consider now the non-relativistic limit of quaternion electrodynamics. First we shall look for an analogue of the Coulomb potential. In the usual case we have $j^0 = \rho = \delta^{(3)}(\boldsymbol{x})$, $\boldsymbol{A} = 0$, $A^0 \sim r^{-1}$. Our analogue will be

$$j^0 = 0 \qquad j^k = \delta(x_0)\mathrm{i}_k \qquad A^0 = 0 \qquad A^k(x_0) = \mathbf{i}\boldsymbol{A}^k(x_0)$$

where j^μ is the current generated by the tachyon moving with the infinite velocity. The Yang–Mills equations look like

$$\partial_0^2\boldsymbol{A}^k + 4g^2(\boldsymbol{A}_k\boldsymbol{A}_j^2 - \boldsymbol{A}_j(\boldsymbol{A}_j\boldsymbol{A}_k)) = g\boldsymbol{j}_k.$$

This equation simplifies for $A^k = \mathrm{i}_k a(x_0)$, $k = 1, 2, 3$ to

$$\partial_0^2 a(x_0) + 8g^2a^3(x_0) = g\delta(x_0).$$

In the approximation $g^2 \to 0$ we have $a(x_0) = \tfrac{1}{2}g|x_0|$, the confining potential in the time variable. Thus the confinement of tachyons is to be expected in quaternion electrodynamics.

The non-relativistic tachyon which interacts with the above field is described by

$$\mathbf{i}\boldsymbol{\partial}\psi + \frac{1}{2m}\partial_0^2\psi = \sum_{k=1}^{3} ga(x_0)\mathrm{i}_k\psi\mathrm{i}_k.$$

But the current j^μ from (3.10) is not conserved now

$$\partial_\mu j^\mu = 2ga(x_0)(\psi^{+2} - \psi^2) = 8ga(x_0)\ \mathrm{Re}\ \psi\ \mathrm{Im}\ \psi.$$

Thus, in the non-relativistic limit, the number of ψ particles is not conserved in the presence of the quaternion electrodynamics field; only the total current (6.1) is conserved. In this respect quaternion electrodynamics differs substantially from the usual electrodynamics and also from the Salam–Weinberg model, where an electron and a neutrino are different particles. In other words, even the static quaternion electromagnetic field will create and annihilate ψ particles.

7. Time–space duality

The time–space duality (TSD) can be set up as the postulate: *fundamental laws of Nature are to be symmetric with respect to time and space* in the sense of the analogy between the quaternion and complex QM described above. This implies either the existence of tachyons dual to the known bradyons or that particles should be massless. The second possibility is, of course, more realistic, especially in the context of the Salam–Weinberg model. *The observed bradyonic character of our world is the consequence of the spontaneous symmetry breaking.* The unbroken massless theory should be TSD symmetric and the interpretation both in CQM and QQM should be possible. The TSD can put some restriction on possible massless theories. Such an argument will be explored in the subsequent paper.

Let us list here the manifestations of TSD:

complex quantum theory	$\leftrightarrow$	quaternion quantum theory
bradyons	$\leftrightarrow$	tachyons
evolution in time	$\leftrightarrow$	evolution in space (§§ 2, 3)
Cauchy data at $\{t = 0\}$	$\leftrightarrow$	Cauchy data at $\{x = 0\}$ (§§ 3, 5)
positivity of an energy	$\leftrightarrow$	positivity of 3-momentum (= left handedness) (§ 3)
straight line lying inside the light cone	$\leftrightarrow$	hyperplane lying outside the light cone (§ 4)
phase invariance $\psi \to \psi\, e^{i\alpha}$	$\leftrightarrow$	phase invariance $\psi \to \psi\, e^{\mathbf{i}\alpha}$ (§ 6)
electrodynamic U(1) gauge field	$\leftrightarrow$	Yang–Mills SU(2) gauge field (§ 6)
energy gap	$\leftrightarrow$	3-momentum gap (§ 5).

We should want to remark that our theory may help in understanding the space-parity breaking. The problem lies in the fact that there is nothing in the principles of quantum theory which distinguishes *a priori* the right-handed and left-handed coordinate system. Our argument goes as follows: Let us consider the theory containing both bradyons and tachyons and let us suppose that both negative energy bradyon states and negative 3-momentum tachyon states are occupied. Such a theory would be *CPT* symmetric but *P* and *T* asymmetric. In this case the orientation of the coordinate system is related to the orientation of time. This argument assumes, of course, that tachyons play an important role in the structure of vacuum. To our mind *tachyons are* *important only in the vacuum sector, because they cannot have any* (usual) *particle-like manifestation.*

Throughout the paper we have found strong interrelations among the following facts:

(i) dimensionality of space–time (one time and three space dimensions),

(ii) topology of the inner and outer parts of the light cone,

(iii) there are two possible quantum mechanics (complex and quaternion ones),

(iv) there are two number fields (complex numbers and quaternions).

The last statement is a purely mathematical (Hurwitz's) theorem asserting that there are exactly two number fields which are associative and contain imaginary units (both properties are necessary in the quantum theory). We left to the reader the evaluation of the philosophical and speculative aspects of such an interrelation (for example, the implication (iv) $\Rightarrow$ (i)).

8. Conclusions

We have shown that quaternion quantum mechanics may be developed along different lines from previous investigations. The resulting theory is intrinsically coherent but a very bizarre one and includes changes in almost all basic assumptions of any physical theory known today. It describes tachyons and it is the $3+1$-dimensional theory of them. If our arguments are true, then the quantum (and also classical) concept of a tachyon must be changed drastically. Tachyons become stranger objects than before. There is a striking analogy called the time–space duality between complex and quaternion quantum mechanics which may be important in the study of massless particles. Another consequence is that the notion of an evolution variable (time in the usual theory) is not logically self-evident and it should result from the 'dynamics' of the theory.

Acknowledgments

An essential part of this work was carried out during a stay at Pisa University in spring 1977; it is the author's pleasure to acknowledge cordial hospitality of Professors G Prodi, M Giaquinta and Dr G Modica. The author also benefited from discussions with Dr M Znojil (UJV Czech. Ac. Sci., Prague) and Dr J Holakovský (FZU Czech. Ac. Sci., Prague). Critical reading of the manuscript by Dr F Smutný (FZU Czech. Ac. Sci., Prague) is also acknowledged. Sincere thanks are due to my brother Dr V Souček (Charles University, Prague) for perpetual interest in this work, invaluable discussions and constructive criticism without which this work would hardly have been carried out.

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